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# Quasiconformal extension of univalent functions and Becker's theorem

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## Abstract

This is a research for a subclass of univalent holomorphic functions on the unit disc normalized by  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ , which can be extended to  $k$ -quasiconformal mappings on the disc  $\{z \mid |z| < R\}$  where  $R > 1$ . Such a subclass is denoted by  $S(k, R)$ . In this note, the class  $S(k, R)$  is introduced through the observation of Becker's theorem which ensures a  $k$ -quasiconformal extendibility of univalent holomorphic functions on the disc to the Riemann sphere with Löwner chains.

## 1 Motivation

Let  $D = \{z \mid |z| < 1\}$  and

$S = \{f \mid f \text{ is holomorphic and univalent on } D, f(0) = f'(0) - 1 = 0\},$

$S(k) = \{f \mid f \in S, f \text{ can be extended to a } k\text{-quasiconformal mapping on } \widehat{\mathbb{C}}\},$

$S_0(k) = \{f \mid f \in S(k), \text{ the extended mappings fix } \infty\},$

respectively, where  $k \in [0, 1)$ . The class  $S(k)$  has been studied by numerous authors in connection with the theory of Teichmüller spaces. In those investigations, an interesting method for quasiconformal extension of univalent functions was obtained by Becker ([1], see also [5]) which relies on the *Löwner chains* described by the *Löwner equation*

$$\frac{\partial f(z, t)}{\partial t} = zp(z, t) \frac{\partial f(z, t)}{\partial z} \quad (1)$$

for  $z \in D$  and  $t \in [0, \infty)$ . This equation determines an expanding flow. Here, the function  $f(z, t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$  is holomorphic in  $|z| < 1$  for each  $t \in [0, \infty)$ , absolutely continuous in  $t \in [0, \infty)$  for each  $|z| < r_0$  and satisfies the inequality  $|f(z, t)| \leq K_0 e^t$  ( $|z| < r_0, t \leq 0$ ) for some positive constants  $K_0$  and  $r_0$ . Also a function  $p(z, t)$  is measurable on  $D \times [0, \infty)$ , holomorphic in  $|z| < 1$ , and satisfies  $\operatorname{Re} p(z, t) > 0$  and the partial differential equation (1) for

a.e.  $t$ .

**Theorem 1 ([1]).** *If  $f(z, t)$  is a univalent solution to (1) with  $p(z, t)$  satisfying the condition*

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| \leq k < 1 \quad (2)$$

*then, for each  $t \geq 0$ , the function  $f_t(z) = f(z, t)$  maps  $\mathbb{D}$  onto a Jordan domain bounded by a  $k$ -quasiconformal image of  $\partial\mathbb{D}$ , and the map  $\hat{f}(z)$  defined by*

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta}, 0) & r \leq 1 \\ f(e^{i\theta}, \log r) & r > 1 \end{cases}$$

*is a  $k$ -quasiconformal extension of  $f(z, 0)$  onto  $\widehat{\mathbb{C}}$  with  $\hat{f}(\infty) = \infty$  (thus  $\hat{f}(z) \in S_0(k)$ ).*

Observe that  $p(\mathbb{D}, t)$  must be contained in the disc  $|z - (1 + k^2)/(1 - k^2)| \leq 2k/(1 - k^2)$  for all  $t \in [0, \infty)$  so that we can apply Theorem 1 to the Löwner chains (Fig.1). This strong assumption can be weakened by restricting the range of the parameter  $t$ . In fact, the following is true;

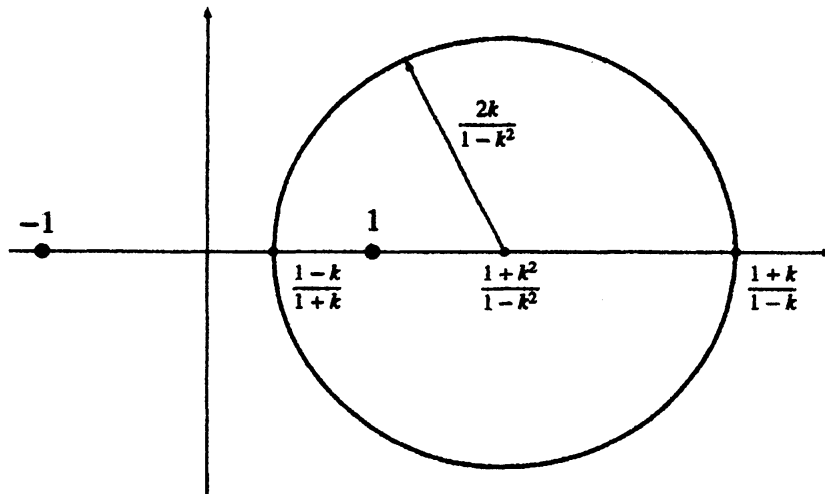


Figure 1 :  $p(z, t)$  must be in this circle for all  $z \in \mathbb{D}$  and  $t \in [0, \infty)$ .

**Corollary 2.** *If  $f(z, t)$  is a univalent solution to (1) and there exists  $t_0 > 0$  such that for all  $t \in [0, t_0]$   $p(z, t)$  satisfies the condition*

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| \leq k < 1, \quad (3)$$

*then the map  $\hat{f}(z)$  is a  $k$ -quasiconformal extension of  $f(z, 0)$  defined on  $\{z \mid |z| < e^{t_0}\}$  with  $\hat{f} \neq \infty$ .*

Now we shall introduce the classes  $S(k, R)$  and  $S_0(k, R)$ ; namely

$$S(k, R) = \{f \mid f \in S, f \text{ can be extended to a } k\text{-quasiconformal mapping } \hat{f} \text{ on } \{|z| < R\}\}$$

and

$$S_0(k, R) = \{f \mid f \in S(k, R), \text{ the extended mapping } \hat{f} \text{ doesn't take } \infty \text{ on } \{|z| < R\}\}$$

respectively, where  $R > 1$ .

## 2 Properties of the class $S(k, R)$

The class  $S(k, R)$  was studied by some authors in another context. We shall give some known results for the classes  $S(k, R)$  and  $\Sigma(k, r)$ , where  $\Sigma(k, r)$  is a family of univalent holomorphic functions on  $\{z \in \widehat{\mathbb{C}} - \overline{\mathbb{D}}\}$  which can be extended to a  $k$ -quasiconformal mapping on  $\{|z| > r\}$ ,  $r < 1$ .

McLeavey [8] (see also [9]) first considered the subclass of  $\Sigma$  with  $K(|z|)$ -quasiconformal extensions into the interior of  $\mathbb{D}$  where  $K(|z|)$  is a piecewise continuous function of bounded variation on  $[r, 1]$ ,  $0 \leq r < 1$ . She obtained for this class the analogs of the classical Grunsky and Goluzin inequalities and sharp estimates for the coefficients  $b_0$  and  $b_1$  of  $\Sigma(k, r)$  and  $a_2$  of  $S_0(k, R)$  with extremal function as follow;

**Theorem 3 ([8]).** *If  $g \in \Sigma(k, r)$  and  $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$  for  $|z| > 1$ , then*

$$|b_1| \leq \frac{k + r^2}{1 + kr^2}.$$

Equality occurs if and only if

$$g(z) = \begin{cases} z + b_0 + \left( \frac{k + r^2}{1 + kr^2} \right) \frac{e^{i\alpha}}{z} & |z| \geq 1 \\ \left( \frac{1}{1 + kr^2} \right) \left( z + \frac{r^2 e^{i\alpha}}{z} + k e^{i\alpha} \bar{z} + \frac{kr^2}{\bar{z}} \right) & r < |z| \leq 1. \end{cases}$$

**Corollary 4 ([8]).** Suppose  $f \in S(k, R)$  and  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  for  $z \in D$ . Then

$$|a_3 - a_2^2| \leq \frac{1 + kR^2}{k + R^2}.$$

If, in addition, extend mappings do not take  $\infty$  on  $\{|z| < R\}$ , then

$$|a_2| \leq 2 \frac{1 + kR}{k + R}. \quad (4)$$

Kühnau [6] also proved similar results of those through introducing the class  $\Sigma(Q_1, \dots, Q_n)$  of  $K(|z|)$ -quasiconformal mapping of the plane which are conformal on  $\{z; |z| > 1\}$  with a development  $f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$  and which have piecewise bounded dilatation in  $D : K(|z|) \leq Q_i$  ( $Q_i \geq 1$ ) in  $R_i < |z| < R_{i-1}$  ( $i = 1, \dots, n$ ), with  $R_0 = 1$ ,  $R_n = 0$ . Schober [9] mentioned above results in his book, Chap.14. He also gave some more results, for instance, generalized Gronwall's area theorem for  $\Sigma(k, r)$ ;

**Theorem 5 ([9]).** If  $g \in \Sigma(k, r)$  and  $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$  for  $|z| > 1$ , then

$$\sum_{m=1}^{\infty} m |b_m|^2 \leq \left( \frac{k + r^2}{1 + kr^2} \right)^2. \quad (5)$$

Under the more general case, Lehto [7] showed a majorant principle for a holomorphic functional as follow;

Let  $A$  be a domain in  $\widehat{\mathbb{C}}$  which is bounded by a quasicircle,  $B$  be a domain whose closure  $\overline{B} \subset A$ , and  $\mathcal{F}_k$ ,  $0 \leq k < 1$ , be a family of functions which are  $k$ -quasiconformal on  $A$  and conformal on  $\overline{B}$ . Denote by  $\mathcal{F}_1$  the family of all conformal mappings on  $B$ .

We introduce four different normalizations to cover a large number of cases appearing in applications. Let  $z_1, z_2, z_3$  be distinct points of  $B$  and  $\alpha_1, \alpha_2, \alpha_3, \beta$  are complex numbers, the

$\alpha$ 's are different from each other and  $\beta \neq 0$ . The families  $\mathcal{F}_k$  and  $\mathcal{F}_1$  are called normalized if all the functions  $f$  of  $A$  contained in  $\mathcal{F}_k$  or  $\mathcal{F}_1$  have one of the following conditions;

1.  $f(z_i) = \alpha_i$ ,  $i = 1, 2, 3$ ,
2.  $f(z_i) = \alpha_i$ ,  $i = 1, 2$ , and  $f(z) \neq \infty$  in  $A$ ,
3.  $f(z_1) = \alpha_1$ ,  $f'(z_1) = \beta$  and  $f(z) \neq \infty$  in  $A$ ,
4. If  $\infty \in B$ , then  $f(z) - z \rightarrow 0$  as  $z \rightarrow \infty$ .

We shall suppose here  $\mathcal{F}_k$  and  $\mathcal{F}_1$  are normalized. Remark that normalized  $\mathcal{F}_k$  and  $\mathcal{F}_1$  are closed normal families.

Let  $\Psi$  be a holomorphic functional defined on the family  $\mathcal{F}_k$  or  $\mathcal{F}_1$ , i.e.  $\Psi(f) = \omega(f(z_0), f'(z_1), \dots, f^{(n)}(z_n))$ , where  $\omega$  is a complex-valued holomorphic function of the variables  $f^{(i)}(z_i)$ ,  $i = 1, 2, \dots$ , each  $f^{(i)}(z_i)$  being the value at fixed point  $z_i \in B$ .

Set

$$M(k) = \sup_{f \in \mathcal{F}_k} |\Psi(f)|, \quad 0 \leq k \leq 1.$$

Since  $\mathcal{F}_k$  is a closed normal family, there exists an extremal function maximizing  $|\Psi(f)|$  in  $\mathcal{F}_k$ .

**Theorem 6 ([7]).** *For a holomorphic functional in  $\mathcal{F}_k$ ,*

$$M(k) \leq M(1) \frac{k + \frac{M(1)}{M(0)}}{1 + k \frac{M(1)}{M(0)}}. \quad (6)$$

This result contains some coefficient estimates as corollaries; for the class  $\Sigma(k, r)$

$$\max_{\Sigma(k, r)} |b_n| \leq \frac{k + r^{n+1}}{1 + kr^{n+1}} \max_{\Sigma} |b_n|, \quad n = 1, 2, \dots,$$

which imply

$$|b_1| \leq \frac{k + r^2}{1 + kr^2} \quad \text{and} \quad |b_2| \leq \frac{2}{3} \frac{k + r^3}{1 + kr^3}.$$

The Grunsky type inequalities for  $\Sigma(k, r)$  also follow easily from the general inequality (6).

For  $f \in \Sigma$ , let  $A_{mn}$ ,  $m, n = 1, 2, \dots$ , be the numbers determined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m, n=1}^{\infty} A_{mn} z^{-m} \zeta^{-n}.$$

Then for any complex numbers  $x_1, x_2, \dots, x_N$ ,

$$\left| \sum_{m,n}^N A_{mn} x_m x_n \right| \leq \frac{k+r^2}{1+kr^2} \sum_n^N \frac{|x_n|^2}{n} \quad (7)$$

and

$$\sum_n^N n \left| \sum_m^N A_{mn} x_m \right|^2 \leq \left( \frac{k+r^2}{1+kr^2} \right)^2 \sum_{n=1}^N \frac{|x_n|^2}{n}. \quad (8)$$

Remark that (7) and (8) is not sharp (see [8]).

Deiermann treats several similar problems of those in [2] and [3] with the method of extremal length. Recently, Krushkal gives a short mention for those reserches in his survey [4], Chap.6.3.

### 3 Main Results

Now the more applications of Theorem 6 are given to  $S_0(k, R)$  and  $\Sigma(k, r)$  (again remark that these results are not sharp because (7) and (8) is not sharp) ;

**Theorem 7.**

$$\sup_{S_0(k,R)} |a_n| \leq n \frac{1 + kR^{n-1}}{k + R^{n-1}}.$$

**Proof.** Let us take  $\mathcal{F}_k = S_0(k, R)$ , then  $\mathcal{F}_1$  is the well-known class  $\mathcal{S}$ . Choose  $\Psi(f) = a_n$ . Then  $M(1) = n$ , and  $M(0) = n/R^{n-1}$  because  $Rf(z/R) \in \mathcal{S}$  for arbitrary  $f \in \mathcal{F}_0$ . Hence the inequality (6) follow the theorem.  $\square$

**Theorem 8 (Generalized Goluzin inequality).** If  $g \in \Sigma(k, r)$  and  $z_v \in \widehat{\mathbb{C}} - \mathbb{D}$ ,  $\gamma_v \in \mathbb{C}$  ( $v = 1, 2, \dots, n$ ),  $n = 1, 2, \dots$ , then

$$\left| \sum_{\mu} \sum_{\nu} \gamma_{\mu} \gamma_{\nu} \log \frac{g(z_{\mu}) - g(z_{\nu})}{z_{\mu} - z_{\nu}} \right| \leq \frac{k+r^2}{1+kr^2} \sum_{\mu} \sum_{\nu} \gamma_{\mu} \overline{\gamma_{\nu}} \log \frac{1}{1 - (z_{\mu} \overline{z_{\nu}})^{-1}}. \quad (9)$$

**Proof.** We shall apply the inequality (7) with  $x_m = \sum_{v=1}^N \gamma_v z_v^{-m}$ ,  $m = 1, 2, \dots$ . In fact, we

have

$$\begin{aligned} \sum_{\mu} \sum_{\nu} \gamma_{\mu} \gamma_{\nu} \log \frac{g(z_{\mu}) - g(z_{\nu})}{z_{\mu} - z_{\nu}} &= - \sum_{m,n} \sum_{\mu,\nu} A_{mn} \gamma_{\mu} \gamma_{\nu} z_{\mu}^{-m} z_{\nu}^{-n} \\ &= - \sum_{m,n} A_{mn} x_m x_n. \end{aligned}$$

Hence (7) shows that the left-hand side of (9) is

$$\begin{aligned} &\leq \frac{1 + kr^2}{k + r^2} \sum_n \frac{1}{n} |x_n|^2 = \frac{1 + kr^2}{k + r^2} \sum_n \frac{1}{n} \sum_{\mu,\nu} \gamma_{\mu} \bar{\gamma}_{\nu} z_{\mu}^{-k} z_{\nu}^{-k} \\ &= \frac{1 + kr^2}{k + r^2} \sum_{\mu,\nu} \gamma_{\mu} \bar{\gamma}_{\nu} \log \frac{1}{1 - (z_{\mu} \bar{z}_{\nu})^{-1}}. \end{aligned}$$

This completes the proof of the theorem.  $\square$

**Theorem 9.** For  $f \in S_0(k, R)$  and  $z \in \mathbb{D}$ ,

$$\left| \log \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + kR}{k + R} \log \frac{1 + |z|}{1 - |z|}.$$

**Proof.** In (9) let  $n = 2$ ,  $\gamma_1 = 1$ ,  $\gamma_2 = -1$ , then

$$\left| \log \frac{g'(z)g'(\zeta)(z - \zeta)^2}{(g(z) - g(\zeta))^2} \right| \leq \frac{k + r^2}{1 + kr^2} \log \frac{|z\bar{\zeta} - 1|^2}{(|z|^2 - 1)(|\zeta|^2 - 1)} \quad (z, \zeta \in \widehat{\mathbb{C}} - \mathbb{D}). \quad (10)$$

We want to apply (9) to the function  $f \in S_0(k, R)$ . If we put

$$g(\zeta) = 1/\sqrt{f(\zeta^{-2})}, \quad (11)$$

then  $g \in \Sigma(k, 1/\sqrt{R})$ . Since  $g$  is odd function, it follows from (10) with  $z = -\zeta$  that

$$\left| 2 \log \frac{\zeta g'(\zeta)}{g(\zeta)} \right| \leq \frac{k + (1/R)}{1 + k(1/R)} 2 \log \frac{|\zeta|^2 + 1}{|\zeta|^2 - 1} \quad (|\zeta| > 1).$$

If we choose  $z = \zeta^{-2}$  and use (11) we obtain the desire inequality.  $\square$

**Corollary 10.** For  $f \in S_0(k, R)$  and  $z \in \mathbb{D}$ ,

$$\left( \frac{1 - |z|}{1 + |z|} \right)^{(1+kR)/(k+R)} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \left( \frac{1 + |z|}{1 - |z|} \right)^{(1+kR)/(k+R)}.$$



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